CALCULUS III

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Preface

Here are my online notes for my Calculus III course that I teach here at Lamar University. Despite the fact that these are my "class notes", they should be accessible to anyone wanting to learn Calculus III or needing a refresher in some of the topics from the class.

These notes do assume that the reader has a good working knowledge of Calculus I topics including limits, derivatives and integration. It also assumes that the reader has a good knowledge of several Calculus II topics including some integration techniques, parametric equations, vectors, and knowledge of three dimensional space.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn't covered in class.

2. In general I try to work problems in class that are different from my notes. However, with Calculus III many of the problems are difficult to make up on the spur of the moment and so in this class my class work will follow these notes fairly close as far as worked problems go. With that being said I will, on occasion, work problems off the top of my head when I can to provide more examples than just those in my notes. Also, I often don't have time in class to work all of the problems in the notes and so you will find that some sections contain problems that weren't worked in class due to time restrictions.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can't anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I've not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.

Outline

Here is a listing and brief description of the material in this set of notes.

Three Dimensional Space

Introduction

In this chapter we will start taking a more detailed look at three dimensional space (3-D space or $\mathbb{R}^3$). This is a very important topic in Calculus III since a good portion of Calculus III is done in three (or higher) dimensional space.

We will be looking at the equations of graphs in 3-D space as well as vector valued functions and how we do calculus with them. We will also be taking a look at a couple of new coordinate systems for 3-D space.

This is the only chapter that exists in two places in my notes. When I originally wrote these notes all of these topics were covered in Calculus II however, we have since moved several of them into Calculus III. So, rather than split the chapter up I have kept it in the Calculus II notes and also put a copy in the Calculus III notes. Many of the sections not covered in Calculus III will be used on occasion there anyway and so they serve as a quick reference for when we need them.

Here is a list of topics in this chapter.

Curvature – We will determine the curvature of a function in this section.

Velocity and Acceleration – In this section we will revisit a standard application of derivatives. We will look at the velocity and acceleration of an object whose position function is given by a vector function.

Cylindrical Coordinates – We will define the cylindrical coordinate system in this section. The cylindrical coordinate system is an alternate coordinate system for the three dimensional coordinate system.

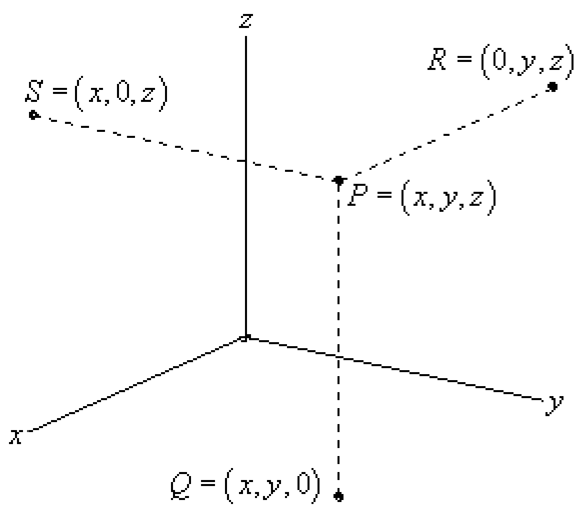
Spherical Coordinates – In this section we will define the spherical coordinate system. The spherical coordinate system is yet another alternate coordinate system for the three dimensional coordinate system.

The 3-D Coordinate System

We'll start the chapter off with a fairly short discussion introducing the 3-D coordinate system and the conventions that we'll be using. We will also take a brief look at how the different coordinate systems can change the graph of an equation.

Let's first get some basic notation out of the way. The 3-D coordinate system is often denoted by $\mathbb{R}^3$. Likewise the 2-D coordinate system is often denoted by $\mathbb{R}^2$ and the 1-D coordinate system is denoted by $\mathbb{R}$. Also, as you might have guessed then a general *n* dimensional coordinate system is often denoted by $\mathbb{R}^n$.

Next, let's take a quick look at the basic coordinate system.



This is the standard placement of the axes in this class. It is assumed that only the positive directions are shown by the axes. If we need the negative axis for any reason we will put them in as needed.

Also note the various points on this sketch. The point *P* is the general point sitting out in 3-D space. If we start at *P* and drop straight down until we reach a *z*-coordinate of zero we arrive at the point *Q*. We say that *Q* sits in the *xy*-plane. The *xy*-plane corresponds to all the points which have a zero *z*-coordinate. We can also start at *P* and move in the other two directions as shown to get points in the *xz*-plane (this is *S* with a *y*-coordinate of zero) and the *yz*-plane (this is *R* with an *x*-coordinate of zero).

Collectively, the *xy*, *xz*, and *yz*-planes are sometimes called the coordinate planes. In the remainder of this class you will need to be able to deal with the various coordinate planes so make sure that you can.

Also, the point *Q* is often referred to as the projection of *P* in the *xy*-plane. Likewise, *R* is the projection of *P* in the *yz*-plane and *S* is the projection of *P* in the *xz*-plane.

Many of the formulas that you are used to working with in $\mathbb{R}^2$ have natural extensions in $\mathbb{R}^3$. For instance the distance between two points in $\mathbb{R}^2$ is given by,

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

While the distance between any two points in $\mathbb{R}^3$ is given by,

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Likewise, the general equation for a circle with center (h, k) and radius *r* is given by,

$$(x - h)^2 + (y - k)^2 = r^2$$

and the general equation for a sphere with center (h, k, l) and radius *r* is given by,

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

With that said we do need to be careful about just translating everything we know about $\mathbb{R}^2$ into $\mathbb{R}^3$ and assuming that it will work the same way. A good example of this is in graphing to some extent. Consider the following example.

Example 1 Graph $x = 3$ in $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$.

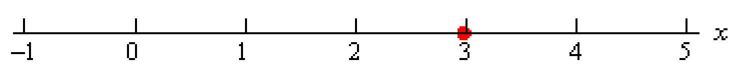
Solution

In $\mathbb{R}$ we have a single coordinate system and so $x = 3$ is a point in a 1-D coordinate system.

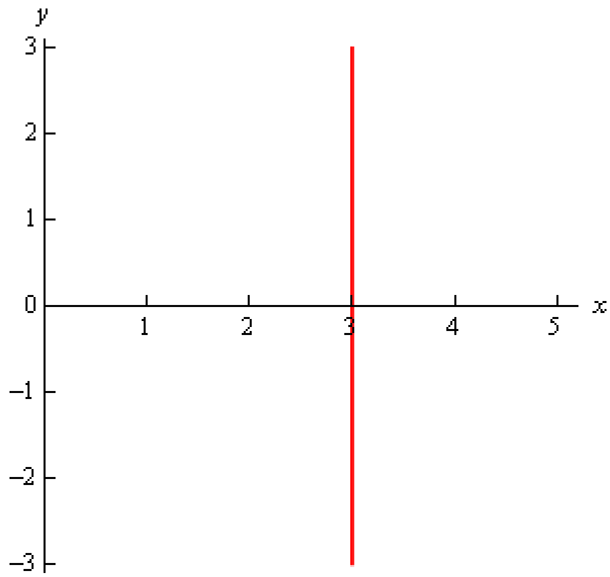
In $\mathbb{R}^2$ the equation $x = 3$ tells us to graph all the points that are in the form $(3, y)$. This is a vertical line in a 2-D coordinate system.

In $\mathbb{R}^3$ the equation $x = 3$ tells us to graph all the points that are in the form $(3, y, z)$. If you go back and look at the coordinate plane points this is very similar to the coordinates for the *yz*-plane except this time we have $x = 3$ instead of $x = 0$. So, in a 3-D coordinate system this is a plane that will be parallel to the *yz*-plane and pass through the *x*-axis at $x = 3$.

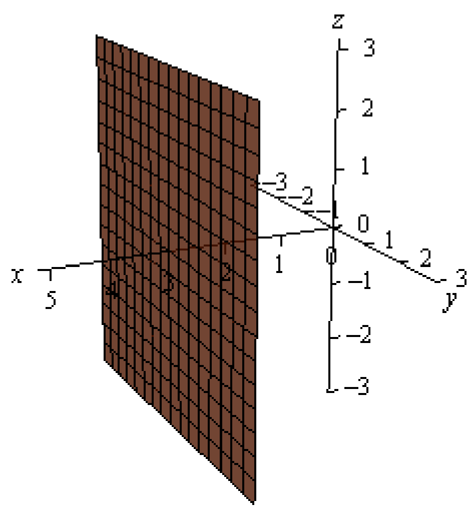
Here is the graph of $x = 3$ in $\mathbb{R}$.

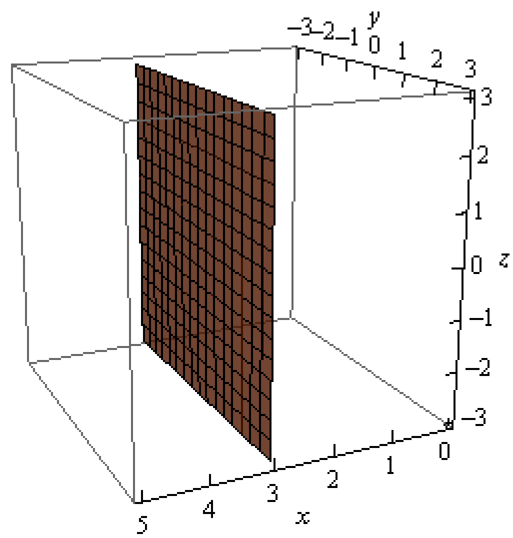


Here is the graph of $x = 3$ in $\mathbb{R}^2$.



Finally, here is the graph of $x = 3$ in $\mathbb{R}^3$. Note that we've presented this graph in two different styles. On the left we've got the traditional axis system and we're used to seeing and on the right we've put the graph in a box. Both views can be convenient on occasion to help with perspective and so we'll often do this with 3D graphs and sketches.





Note that at this point we can now write down the equations for each of the coordinate planes as well using this idea.

$$z = 0 \qquad xy-\text{plane}$$
$$y = 0 \qquad xz-\text{plane}$$
$$x = 0 \qquad yz-\text{plane}$$

Let's take a look at a slightly more general example.

Example 2 Graph $y = 2x - 3$ in $\mathbb{R}^2$ and $\mathbb{R}^3$.

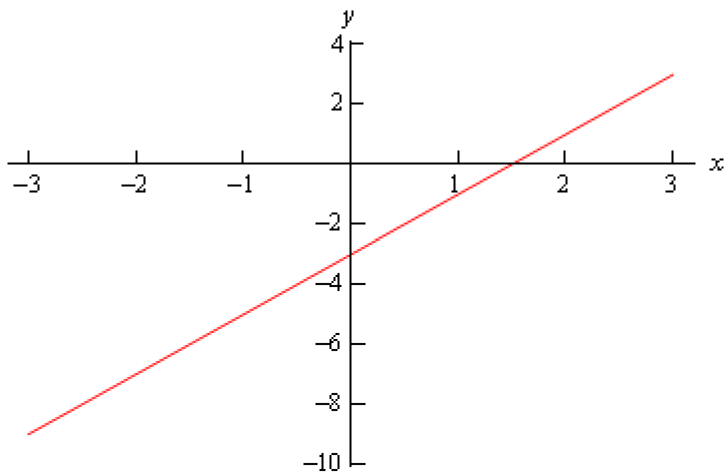
Solution

Of course we had to throw out $\mathbb{R}$ for this example since there are two variables which means that we can't be in a 1-D space.

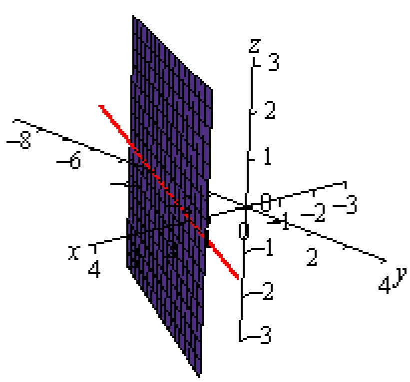
In $\mathbb{R}^2$ this is a line with slope 2 and a *y* intercept of -3.

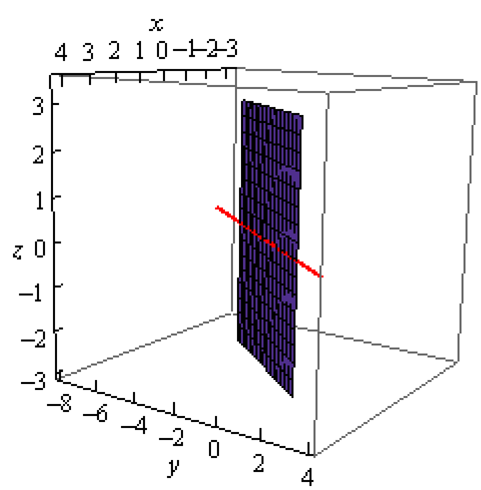
However, in $\mathbb{R}^3$ this is not necessarily a line. Because we have not specified a value of *z* we are forced to let *z* take any value. This means that at any particular value of *z* we will get a copy of this line. So, the graph is then a vertical plane that lies over the line given by $y = 2x - 3$ in the *xy*-plane.

Here is the graph in $\mathbb{R}^2$.



here is the graph in $\mathbb{R}^3$.





Notice that if we look to where the plane intersects the *xy*-plane we will get the graph of the line in $\mathbb{R}^2$ as noted in the above graph by the red line through the plane.

Let's take a look at one more example of the difference between graphs in the different coordinate systems.

Example 3 Graph $x^2 + y^2 = 4$ in $\mathbb{R}^2$ and $\mathbb{R}^3$.

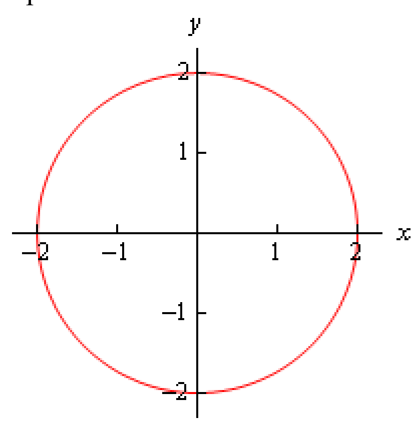
Solution

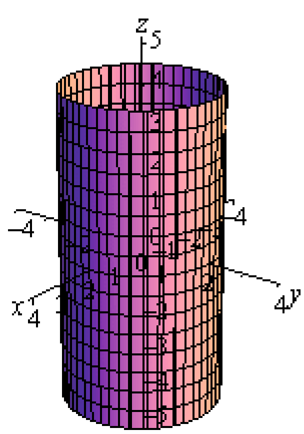
As with the previous example this won't have a 1-D graph since there are two variables.

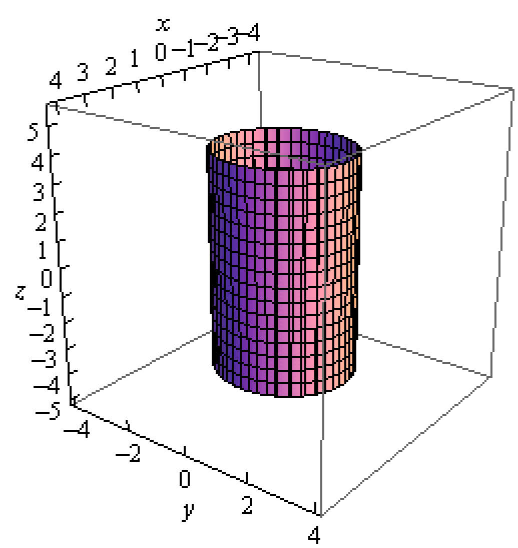
In $\mathbb{R}^2$ this is a circle centered at the origin with radius 2.

In $\mathbb{R}^3$ however, as with the previous example, this may or may not be a circle. Since we have not specified *z* in any way we must assume that *z* can take on any value. In other words, at any value of *z* this equation must be satisfied and so at any value *z* we have a circle of radius 2 centered on the *z*-axis. This means that we have a cylinder of radius 2 centered on the *z*-axis.

Here are the graphs for this example.







Notice that again, if we look to where the cylinder intersects the *xy*-plane we will again get the circle from $\mathbb{R}^2$.

We need to be careful with the last two examples. It would be tempting to take the results of these and say that we can't graph lines or circles in $\mathbb{R}^3$ and yet that doesn't really make sense. There is no reason for there to not be graphs lines or circles in $\mathbb{R}^3$. Let's think about the example of the circle. To graph a circle in $\mathbb{R}^3$ we would need to do something like $x^2 + y^2 = 4$ at $z = 5$. This would be a circle of radius 2 centered on the *z*-axis at the level of $z = 5$. So, as long as we specify a *z* we will get a circle and not a cylinder. We will see an easier way to specify circles in a later section.

We could do the same thing with the line from the second example. However, we will be looking at line in more generality in the next section and so we'll see a better way to deal with lines in $\mathbb{R}^3$ there.

The point of the examples in this section is to make sure that we are being careful with graphing equations and making sure that we always remember which coordinate system that we are in.

Another quick point to make here is that, as we've seen in the above examples, many graphs of equations in $\mathbb{R}^3$ are surfaces. That doesn't mean that we can't graph curves in $\mathbb{R}^3$. We can and will graph curves in $\mathbb{R}^3$ as well as we'll see later in this chapter.

Equations of Lines

In this section we need to take a look at the equation of a line in $\mathbb{R}^3$. As we saw in the previous section the equation $y = mx + b$ does not describe a line in $\mathbb{R}^3$, instead it describes a plane. This doesn't mean however that we can't write down an equation for a line in 3-D space. We're just going to need a new way of writing down the equation of a curve.

So, before we get into the equations of lines we first need to briefly look at vector functions. We're going to take a more in depth look at vector functions later. At this point all that we need to worry about is notational issues and how they can be used to give the equation of a curve.

The best way to get an idea of what a vector function is and what its graph looks like is to look at an example. So, consider the following vector function.

$$\vec{r}(t) = \langle t, 1 \rangle$$

A vector function is a function that takes one or more variables, one in this case, and returns a vector. Note as well that a vector function can be a function of two or more variables. However, in those cases the graph may no longer be a curve in space.

The vector that the function gives can be a vector in whatever dimension we need it to be. In the example above it returns a vector in $\mathbb{R}^2$. When we get to the real subject of this section, equations of lines, we'll be using a vector function that returns a vector in $\mathbb{R}^3$

Now, we want to determine the graph of the vector function above. In order to find the graph of our function we'll think of the vector that the vector function returns as a position vector for points on the graph. Recall that a position vector, say $\vec{v} = \langle a, b \rangle$, is a vector that starts at the origin and ends at the point (a, b).

So, to get the graph of a vector function all we need to do is plug in some values of the variable and then plot the point that corresponds to each position vector we get out of the function and play connect the dots. Here are some evaluations for our example.

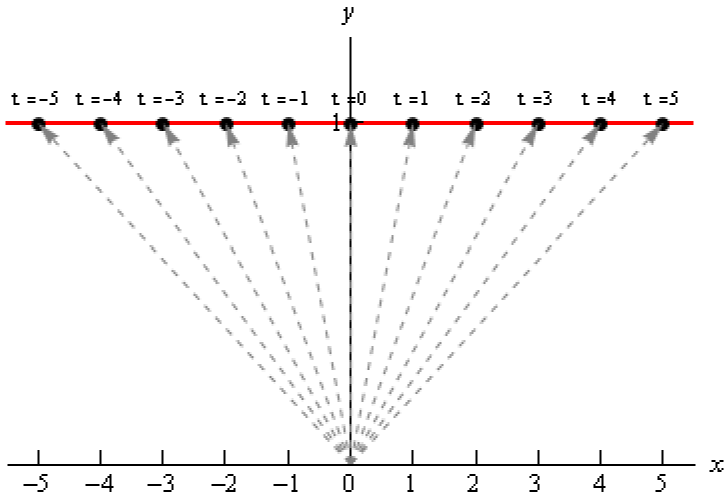
$$\vec{r}(-3) = \langle -3, 1 \rangle \qquad \vec{r}(-1) = \langle -1, 1 \rangle \qquad \vec{r}(2) = \langle 2, 1 \rangle \qquad \vec{r}(5) = \langle 5, 1 \rangle$$

So, each of these are position vectors representing points on the graph of our vector function. The points,

$$(-3, 1) \qquad (-1, 1) \qquad (2, 1) \qquad (5, 1)$$

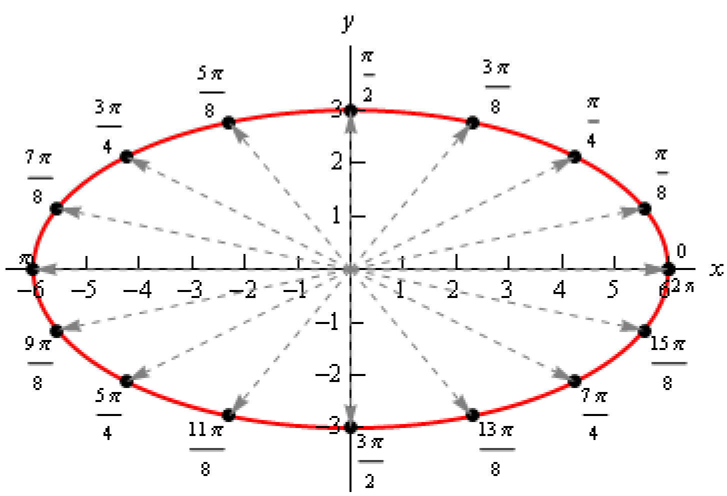
are all points that lie on the graph of our vector function.

If we do some more evaluations and plot all the points we get the following sketch.



In this sketch we've included the position vector (in gray and dashed) for several evaluations as well as the *t* (above each point) we used for each evaluation. It looks like, in this case the graph of the vector equation is in fact the line $y = 1$.

Here's another quick example. Here is the graph of $\vec{r}(t) = \langle 6\cos t, 3\sin t \rangle$.



In this case we get an ellipse. It is important to not come away from this section with the idea that vector functions only graph out lines. We'll be looking at lines in this section, but the graphs of vector function do not have to be lines as the example above shows.

We'll leave this brief discussion of vector function with another way to think of the graph of a vector function. Imagine that a pencil/pen is attached to the end of the position vector and as we increase the variable the resulting position vector moves and as it moves the pencil/pen on the end sketches out the curve for the vector function.

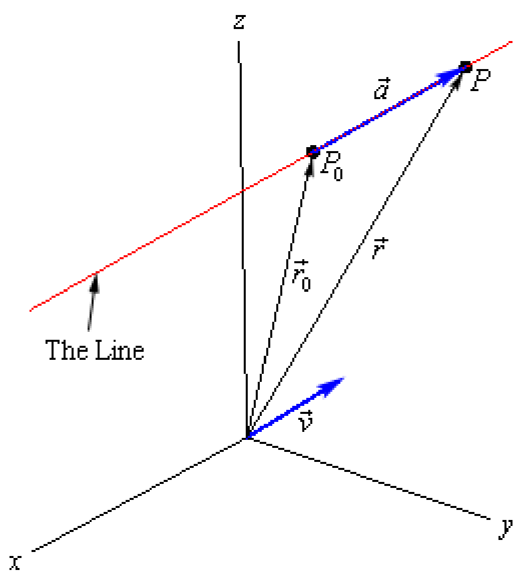
Okay, we now need to move into the actual topic of this section. We want to write down the equation of a line in $\mathbb{R}^3$ and as suggested by the work above we will need a vector function to do this. To see how we're going to do this let's think about what we need to write down the equation of a line in $\mathbb{R}^2$. In two dimensions we need the slope (*m*) and a point that was on the line in order to write down the equation.

In $\mathbb{R}^3$ that is still all that we need except in this case the "slope" won't be a simple number as it was in two dimensions. In this case we will need to acknowledge that a line can have a three dimensional slope. So, we need something that will allow us to describe a direction that is potentially in three dimensions. We already have a quantity that will do this for us. Vectors give directions and can be three dimensional objects.

So, let's start with the following information. Suppose that we know a point that is on the line, $P_0 = (x_0, y_0, z_0)$, and that $\vec{v} = \langle a, b, c \rangle$ is some vector that is parallel to the line. Note, in all likelihood, $\vec{v}$ will not be on the line itself. We only need $\vec{v}$ to be parallel to the line. Finally, let $P = (x, y, z)$ be any point on the line.

Now, since our "slope" is a vector let's also represent the two points on the line as vectors. We'll do this with position vectors. So, let $\vec{r_0}$ and $\vec{r}$ be the position vectors for P_0 and P respectively. Also, for no apparent reason, let's define $\vec{a}$ to be the vector with representation $\overrightarrow{P_0P}$.

We now have the following sketch with all these points and vectors on it.



Now, we've shown the parallel vector, $\vec{v}$, as a position vector but it doesn't need to be a position vector. It can be anywhere, a position vector, on the line or off the line, it just needs to be parallel to the line.

Next, notice that we can write $\vec{r}$ as follows,

$$\vec{r} = \vec{r_0} + \vec{a}$$

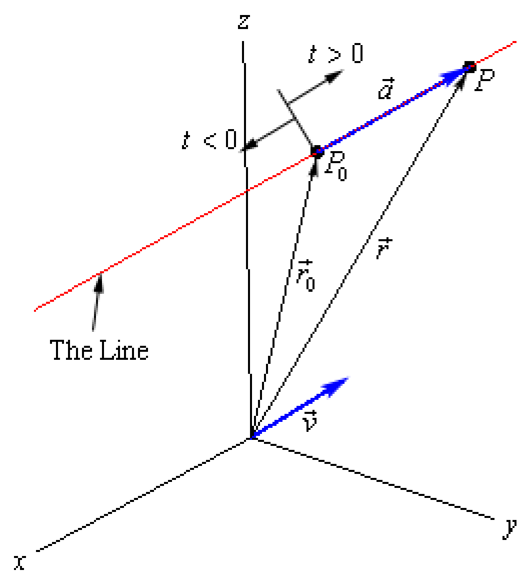
If you're not sure about this go back and check out the sketch for vector addition in the vector arithmetic section. Now, notice that the vectors $\vec{a}$ and $\vec{v}$ are parallel. Therefore there is a number, *t*, such that

$$\vec{a} = t\,\vec{v}$$

We now have,

$$\vec{r} = \overrightarrow{r_0} + t\,\vec{v} = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$

This is called the **vector form of the equation of a line**. The only part of this equation that is not known is the t. Notice that $t\,\vec{v}$ will be a vector that lies along the line and it tells us how far from the original point that we should move. If t is positive we move away from the original point in the direction of $\vec{v}$ (right in our sketch) and if t is negative we move away from the original point in the opposite direction of $\vec{v}$ (left in our sketch). As t varies over all possible values we will completely cover the line. The following sketch shows this dependence on t of our sketch.



There are several other forms of the equation of a line. To get the first alternate form let's start with the vector form and do a slight rewrite.

$$\vec{r} = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$
$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

The only way for two vectors to be equal is for the components to be equal. In other words,

$$x = x_0 + ta$$
$$y = y_0 + tb$$
$$z = z_0 + tc$$

This set of equations is called the **parametric form of the equation of a line**. Notice as well that this is really nothing more than an extension of the parametric equations we've seen previously. The only difference is that we are now working in three dimensions instead of two dimensions.

To get a point on the line all we do is pick a t and plug into either form of the line. In the vector form of the line we get a position vector for the point and in the parametric form we get the actual coordinates of the point.

There is one more form of the line that we want to look at. If we assume that a, b, and c are all non-zero numbers we can solve each of the equations in the parametric form of the line for t. We can then set all of them equal to each other since t will be the same number in each. Doing this gives the following,

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$$

This is called the **symmetric equations of the line**.

If one of a, b, or c does happen to be zero we can still write down the symmetric equations. To see this let's suppose that $b=0$. In this case t will not exist in the parametric equation for y and so we will only solve the parametric equations for x and z for t. We then set those equal and acknowledge the parametric equation for y as follows,

$$\frac{x-x_0}{a}=\frac{z-z_0}{c} \qquad\qquad y=y_0$$

Let's take a look at an example.

Example 1 Write down the equation of the line that passes through the points $(2,-1,3)$ and $(1,4,-3)$. Write down all three forms of the equation of the line.

Solution

To do this we need the vector $\vec{v}$ that will be parallel to the line. This can be any vector as long as it's parallel to the line. In general, $\vec{v}$ won't lie on the line itself. However, in this case it will. All we need to do is let $\vec{v}$ be the vector that starts at the second point and ends at the first point. Since these two points are on the line the vector between them will also lie on the line and will hence be parallel to the line. So,

$$\vec{v}=\langle 1,-5,6\rangle$$

Note that the order of the points was chosen to reduce the number of minus signs in the vector. We could just have easily gone the other way.

Once we've got $\vec{v}$ there really isn't anything else to do. To use the vector form we'll need a point on the line. We've got two and so we can use either one. We'll use the first point. Here is the vector form of the line.

$$\vec{r}=\langle 2,-1,3\rangle+t\langle 1,-5,6\rangle=\langle 2+t,-1-5t,3+6t\rangle$$

Once we have this equation the other two forms follow. Here are the parametric equations of the line.