On the mechanism of sound production in organ pipes*

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It is shown that both the Cremer-Ising and Coltman mechanisms for sound production in organ pipes are comprehended by a more general approach, based on conservation of linear momentum. By calculating force per unit area exerted by the jet on a control volume containing the mixing region, and equating this to the difference in pressure along the pipe axis, it is possible to derive an expression for acoustic particle velocity in the standing wave as a function of the jet driving flow spectrum. The momentum model of the jet-pipe interaction is able to explain the Coltman radiation symmetry effect, and also accounts for the role of entrained air in sound production. Additional spectral interaction terms, not previously noted, are found to play a significant role in the production of sound-pressure fluctuations in the pipe. The fluctuating lift force at the edge is found to contribute to the sustenance of the pipe-cavity oscillation below resonance, opposing it above resonance. In the near vicinity of the resonant frequency, edgetone effects are relatively small.

Subject Classification: 6.5.

INTRODUCTION

The mechanism of organ pipe sound generation is of some historic interest as one of the "unsolved" problems of classical physics. The computation of normal mode frequencies of the pipe is straightforward enough and much good early work was done on things like end correction. Nevertheless, the actual process by which the jet and resonator interact to produce sound has long remained a mystery. Helmholtz believed the essence of the problem to be the introduction of fluctuating jet fluid into the resonant cavity during periods of rarefaction, causing acoustic condensation.[2] Rayleigh[1] pointed out that this mechanism would be most effective at locations along a standing wave where the density fluctuations are maximum, that is, at velocity nodes, whereas at velocity antinodes the jet would tend to drive the cavity fluid via tangential acceleration. Since the mouth of an organ pipe is closer to an antinode than a node, he supposed that the latter mechanism is the predominant one. He attempted no mathematical formulation of the problem, however, concluding that ". . . for a fuller explanation we must probably await a better knowledge of the mechanics of jets." Actually Rayleigh is also largely responsible for developing the theory which led to some of the "better knowledge" of jets necessary to complete the solution. In Chapter 21 of his *Theory of Sound* he laid the foundation for the theory of jet instability, remarking that a solution of this problem is of such importance ". . . as to demand all the consideration we can give." An extension of Rayleigh's approach, the Orr–Sommerfeld equation, was later successfully used to account for the growth of instability in boundary layers and jets.[3–5] Progress on organ pipe theory since Rayleigh has been slow. Many workers have been preoccupied with theories of cast-off vortices and the closely related edgetone phenomenon. In fact, it has been commonly assumed that pipe sound is simply a coupling of edgetone oscillation to a pipe resonator.[6] Mercer,[7] in an article on organ pipe voicing, wisely cautioned against trying to explain pipe mechanism as merely an edgetone effect, since (among other reasons) organ pipe blowing frequencies are typically lower than the expected edgetone frequency. Nevertheless, the intimate relation between "pipe" tones and edgetones is such that progress in the study of one inevitably influenced the understanding of the other. A careful experimental study by Brown[8,9] in the late thirties resulted in an empirical formula for predicting edgetone frequencies. In 1954, using a general integral equation approach, Nyborg was able to compute jet-edge oscillating profiles for an underwater edgetone system.[10,11] A physical model for the mechanism of edgetone production was developed in the early sixties by Powell.[12,13] He showed that the feedback required to sustain the oscillation comes from the fluctuating lift force at the edge, caused by the impinging jet. At that time he suggested that a similar mechanism might be responsible for resonator control of the oscillation in organ pipes, with primary feedback via the resonator rather than the edge.

Following through the implications of Powell's idea Cremer,[14] Ising,[15] and Bechert[16] developed a conceptual model of the organ pipe as a positive feedback amplifier system, separating the mechanism of disturbance amplification (jet waves) from the mechanism of positive feedback (acoustic resonance). In their original papers delivered at the 5th International Congress on Acoustics in 1965, the coupling between the jet and the standing-wave field of the resonator was assumed to be via the acoustic condensation model of Helmholtz, while the feedback from pipe to jet was taken to be a simple transverse convection of the jet fluid in the standing wave. Feedback from the fluctuating lift force at the edge was ignored. Later (1968) Cremer[17] and Ising[18] refined their model to allow for jet instability and the effect of entrained air carried along by the jet.

The Cremer–Ising theory accounts qualitatively for several distinctive features of organ pipe oscillation, namely:

(1) the association of the blown frequency with organ pipe modes,

(2) the tendency of the fundamental pipe frequency to rise with blowing pressure, the pipe reaching its peak sound intensity at a frequency slightly sharp of the unblown mode, and

(3) the tendency to excite higher pipe modes with sufficient driving pressure.

They were able to get fair quantitative agreement with experimental data for a softly voiced pipe, though it was necessary to obtain some of the parameters of the theory empirically. Their model is not designed to take spectral interaction into account and would not be expected to work well for rich-sounding pipes such as the diapason.

Objections to the model of Cremer and Ising were raised by Coltman,[19] who found it difficult to accept the idea that the jet can cause pipe excitation in the manner of a simple source at the mouth. His own approach was to account for the jet drive as a sort of pumping mechanism by which the jet produces a fluctuating *pressure gradient* due to incompressible reaction to shear forces created by its own decay. (This is similar to the mechanism which had been suggested vaguely by Rayleigh.) To illustrate the difference between the two models of pipe drive, circuit analogies are given in Fig. 1. For the Cremer–Ising drive mechanism, shown schematically in Fig. 1(a), the jet "sees" the pipe as a parallel-resonant circuit consisting of the mouth branch of impedance Z_{m1} and a pipe branch of impedance Z_{p1} which meet at the driving point. The sound pressure produced locally by the jet is thus the product of the jet volume flow component Q_{j1} (assumed to be the first Fourier component of the fluctuating jet flow into the pipe) and the parallel acoustic impedance Z_{d1} evaluated at the driving point:

$$p_{m1}=Q_{j1}Z_{d1}=Q_{j1}[Z_{p1}Z_{m1}/(Z_{p1}+Z_{m1})]. \quad (1)$$

In Coltman's model, represented by the circuit analogy of Fig. 1(b), the resonator becomes series driven, the jet supplying a pressure "emf"

$$\Delta p_1 \doteq (S_j/S_p)\rho V_j^2 \quad (2)$$

due to its loss of momentum in expanding from orifice area, S_j, to pipe area, S_p, where V_j is the initial jet speed. The resulting local pressure at the mouth is given by the product of the mouth volume flow, Q_{m1} (i.e., the volume flow of the standing wave) and the

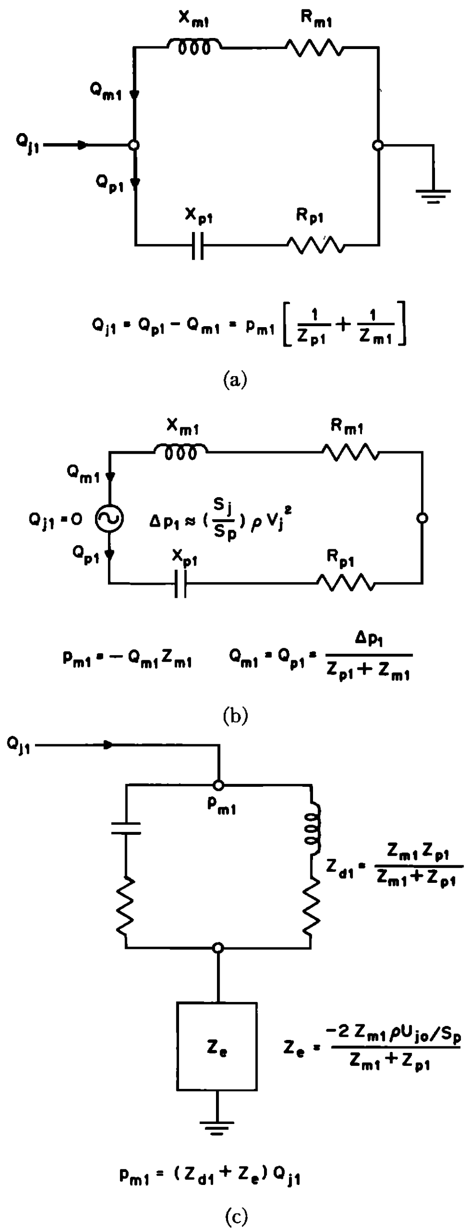

FIG. 1. (a) Equivalent circuit for parallel resonator drive model of Cremer and Ising; jet volume flow, Q_{j1}, includes effect of entrained air. (b) Equivalent circuit for series resonator drive model of Coltman. (c) Approximate equivalent circuit for momentum model of Elder. This is strictly applicable only for sinusoidal excitation.

mouth branch impedance:

$$p_{m1}=-Q_{m1}Z_{m1}, \quad (3)$$

where the negative sign is used consistent with the convention of positive inward flow. In terms of the differential pressure, Δp_1, this may be written[20]:

$$p_{m1}=\Delta p_1[-Z_{m1}/(Z_{p1}+Z_{m1})]. \quad (4)$$

Note that for Coltman's model, $Q_{j1}=0$.

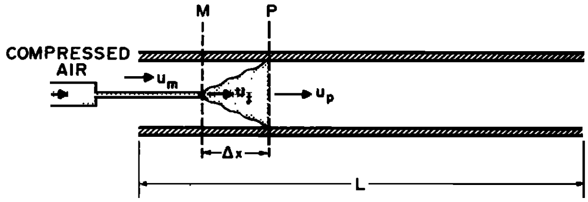

FIG. 2. Schematic representation of momentum model of organ pipe drive, showing control volume, MP, over which turbulent mixing takes place.

In support of his point of view, Coltman did an experiment to show that the source strength at the two ends of a symmetrically constructed open pipe are equal.[21] From this he concluded that there could be no net simple source action of the jet. His claim for this as the sole explanation of the effect has been challenged by Elder,[22] however, who showed that both the Cremer–Ising and Coltman mechanisms are comprehended by a more general approach to the problem, based on conservation of linear momentum, and that equal source strengths at the two ends are to be expected for an open pipe.

It is the purpose of the present paper to develop this approach in some detail, showing how the collision of the jet fluid with the standing wave field of the resonator gives rise to the rich spectrum associated with organ pipe sound. The result of the analysis is a set of equations for the Fourier components of the acoustic flow in the pipe in terms of the Fourier components of the jet drive and the time, i.e., the complex "acoustic gain" of the system. Not dealt with in the present paper is the complementary analysis of the modulation of the jet centerline profile by the field in the pipe, resulting in a set of equations for jet drive components in terms of pipe flow and time. The analysis of the effect of pipe on jet gives the mathematical description of the feedback process required to complete the "circuit." Generalizing Cremer's root-locus method, the simultaneous solution of these two sets of equations defines all of the acceptable loci in solution space for stable oscillating conditions. This procedure, which is best done by an iterative computer technique on account of the several nonlinearities involved,[22] will be treated in a later paper.

I. MOMENTUM MODEL OF JET DRIVE

A. Derivation of the General Momentum Equation

To understand how a fluctuating jet can drive a complex standing wave in a pipe resonator, it is helpful to consider a simple one-dimensional model that has many of the essential features of the organ pipe problem. In Fig. 2 is shown a long, narrow, open pipe of constant cross section S_p and length L (including appropriate end correction for acoustic standing waves). Now suppose a thin tube of cross section S_j to be inserted into the pipe a distance x along the axis (from the left). By means of this tube, a jet of air is introduced into the pipe in the axial direction at a point near one end. If the volume flow of injected air is Q_j, the average linear speed of the jet will be $U_j=Q_j/S_j$. For sufficiently great speed, the flow will be turbulent and over a short distance, Δx, complete mixing with the pipe fluid will occur. Let the mixing region be designated by planes M and P through the locations x and $x+\Delta x$, respectively. Clearly the mixing process must accelerate pipe fluid to the right, up to some average speed $U_p=Q_p/S_p$, where Q_p is the pipe volume flow, as the jet fluid is slowed down to this speed at plane P. Also, the removal of pipe fluid from the cylindrical region MP must cause a dropping of static pressure at plane M, drawing in fresh pipe fluid from the left end at average speed U_m. Letting S_m be the effective pipe cross section in the region penetrated by the tube, the volume flow into the left end of the pipe is $Q_m=U_mS_m$. For acoustic flow the average values U_p and U_m are very nearly equal to their axial values, and in what follows we will ignore the distinction between average and axial values of these velocities.

Now suppose that the jet flow is modulated by some external valving action in periodic fashion. Specifically, let the jet volume flow rate be given by the expression

$$Q_j=Q_{j0}+\sum_{n=1}^{\infty}Q_{jn}\cos(n\omega t+\phi_{jn}). \tag{5}$$

This will cause a disturbance of fundamental frequency $f=\omega/2\pi$ and wavelength λ to be present in the pipe.

If most of the mixing process takes place away from the walls of the pipe, the tangential stress due to skin friction will be small and the loss of jet momentum in MP will result mainly in the simple pressure rise from p_m at plane M to p_p at plane P. The net force on the control volume MP due to this pressure difference can then be equated to the rate at which momentum is building up in MP:

$$-\rho\dot{Q}_p\Delta x+(M_m-M_p)=(p_p-p_m)S_p, \tag{6}$$

where M_p and M_m are the rates of convection of linear momentum past planes P and M, respectively. The momentum of the fluid at P is being convected at speed U_p, so that

$$M_p=(\rho U_pS_p)U_p, \tag{7}$$

while the rate of momentum flow through M has two parts:

$$M_m=(\rho U_jS_j)U_j+(\rho U_mS_m)U_m. \tag{8}$$

Finally, if $\Delta x/\lambda\ll 1$, we may take the flow through MP to be approximately incompressible, giving

$$U_pS_p=U_jS_j+U_mS_m. \tag{9}$$

Substituting Eqs. 7, 8, and 9 in Eq. 6,

$$\rho\Delta x(\dot{U}_jA_{jp}+\dot{U}_mA_{mp})+(p_p-p_m)=\rho U_j^2A_{jp}(1-A_{jp})+\rho U_m^2A_{mp}(1-A_{mp})-2\rho U_jU_mA_{mp}A_{jp}, \quad (10)$$

where $A_{jp}=S_j/S_p$ and $A_{mp}=S_m/S_p$. Now if we associate the fluctuations in U_m and U_p with longitudinal acoustic waves in the pipe, then the pressure at M will, in general, be made up of a dc term, p_{m0}, and the sum of the partial acoustic pressures due to the Fourier components of flow contained in U_m. Likewise, the pressure at P will be made up of a dc term, p_{p0}, plus a sum of partial pressures due to the Fourier components of U_p. In phasor notation:

$$\mathbf{p}_m=p_{m0}-\sum_{n=1}\mathbf{Z}_{mn}\mathbf{U}_{mn}S_m, \quad (11)$$
$$\mathbf{p}_p=p_{p0}+\sum_{n=1}\mathbf{Z}'_{pn}\mathbf{U}_{pn}S_p,$$

where

$$\mathbf{U}_{mn}=U_{mn}\exp[j(n\omega t+\phi_{mn})],$$
$$\mathbf{U}_{pn}=U_{pn}\exp[j(n\omega t+\phi_{pn})]=\mathbf{U}_{jn}A_{jp}+\mathbf{U}_{mn}A_{mp}, \quad (12)$$
$$\mathbf{U}_{jn}=\mathbf{Q}_{jn}/S_j=U_{jn}\exp[j(n\omega t+\phi_{jn})]$$

and where $\mathbf{Z}_{mn}$ and $\mathbf{Z}_{pn}'$ are the values of the complex acoustic impedances at the nth harmonic evaluated along the axis of the pipe (looking outward from the control volume) at planes M and P, respectively; i.e., $\mathbf{Z}_{mn}=\mathbf{Z}_m(n\omega)$, $\mathbf{Z}_{pn}'=\mathbf{Z}_p'(n\omega)$. The resulting momentum conservation equation is then

$$p_{p0}-p_{m0}+\sum_{n=1}\mathrm{Re}[\mathbf{Z}'_{pn}(A_{mp}\mathbf{U}_{mn}+A_{jp}\mathbf{U}_{jn})S_p+\mathbf{Z}_{mn}\mathbf{U}_{mn}S_m+j\rho ck\Delta xn(\mathbf{U}_{jn}A_{jp}+\mathbf{U}_{mn}A_{mp})]$$
$$=\rho A_{jp}(1-A_{jp})(\mathrm{Re}\sum_{n=0}U_{jn})^2+\rho A_{mp}(1-A_{mp})(\mathrm{Re}\sum_{n=0}U_{mn})^2-2\rho A_{mp}A_{jp}(\mathrm{Re}\sum_{n=0}U_{jn})(\mathrm{Re}\sum_{n=0}U_{mn}). \quad (13)$$

Owing to the orthogonality of the Fourier Series expansion, substitution of Eqs. 12 into Eq. 13 leads to an infinite set of equations relating the $\mathbf{U}_{jn}$ and the $\mathbf{U}_{mn}$, one for each of the Fourier components. These will be considered separately. The above approach, although it makes use of the "linear" concepts of Fourier analysis and impedance, is valid in this case because we have required the drive to be periodic. Benade and Gans,[23] who pioneered the non-differential-equation approach to nonlinear systems, have shown that this procedure is also applicable to the case of self-excited oscillation, whenever steady-state periodic solutions are realized.[24]

B. Zero-Order Equation

Collecting terms of zero order in the Fourier expansion represented by Eq. 13, i.e., the time average values, gives

$$p_{p0}-p_{m0}=\rho A_{jp}[(1-A_{jp})U_{j0}^2+A_{mj}(1-A_{mp})U_{m0}^2-2A_{mp}U_{j0}U_{m0}]+\sum_{n=1}\tfrac{1}{2}\rho A_{jp}[(1-A_{jp})U_{jn}^2+A_{mj}(1-A_{mp})U_{mn}^2-2A_{mp}U_{jn}U_{mn}\cos(\phi_{jn}-\phi_{mn})], \quad (14)$$

where $A_{mj}=S_m/S_j$. This tells us that there will be a dc thrust on the control volume, due principally to loss of dynamic pressure by the jet. The strength of the effect is reduced by the area ratio, since the jet does not initially fill the pipe cross section. The pressure drop across MP is augmented by rectification of the fluctuating components of velocity and by additional contributions due to interactions between jet and pipe flow components of same order.

C. Equation for the Fundamental

The equation relating terms of first order in the Fourier expansion is the following:

$$S_m\mathbf{U}_{m1}=(\mathbf{Q}_{m1})_I+(\mathbf{Q}_{m1})_{II}+(\mathbf{Q}_{m1})_{III}, \quad (15)$$
$$(\mathbf{Q}_{m1})_I=[-\mathbf{Z}_{p1}/(\mathbf{Z}_{s1}+\mathbf{R}_j)]\mathbf{Q}_{j1}, \quad (16a)$$
$$(\mathbf{Q}_{m1})_{II}=[(2\rho/S_p)/(\mathbf{Z}_{s1}+\mathbf{R}_j)]\times\{(1-A_{jp})[U_{j0}+\tfrac{1}{2}U_{j2}\exp(\phi_{j2}-2\phi_{j1})]-A_{mp}[U_{m0}+\tfrac{1}{2}U_{m2}\exp j(\phi_{m2}-2\phi_{j1})]\}\mathbf{Q}_{j1}, \quad (16b)$$
$$(\mathbf{Q}_{m1})_{III}=[\rho A_{jp}e^{j\omega t}/(\mathbf{Z}_{s1}+\mathbf{R}_j)]\times\sum_{n=2}[\mathbf{a}U_{jn}U_{j(n+1)}+\mathbf{b}U_{mn}U_{m(n+1)}+\mathbf{c}U_{jn}U_{m(n+1)}+\mathbf{d}U_{j(n+1)}U_{mn}], \quad (16c)$$

and where

$$\mathbf{Z}_{s1}=\mathbf{Z}_{m1}+\mathbf{Z}_{p1}$$
$\equiv$ combined series impedance of resonator,
$$\mathbf{Z}_{p1}=\mathbf{Z}_{p1}'+j\rho ck\Delta x/S_p$$
$\equiv$ pipe impedance evaluated at plane M,
$$\mathbf{R}_j=(2\rho A_{jp}/S_p)\{U_{j0}+\tfrac{1}{2}U_{j2}\times\exp j(\phi_{j2}-2\phi_{m1})-[(1-A_{mp})/A_{jp}]\times[U_{m0}+\tfrac{1}{2}U_{m2}\exp j(\phi_{m2}-2\phi_{m1})]\}, \quad (17)$$
$$\mathbf{a}=(1-A_{jp})\exp j[\phi_{j(n+1)}-\phi_{jn}],$$
$$\mathbf{b}=A_{mj}(1-A_{mp})\exp j[\phi_{m(n+1)}-\phi_{mn}],$$
$$\mathbf{c}=-A_{mp}\exp j[\phi_{m(n+1)}-\phi_{jn}],$$
$$\mathbf{d}=-A_{mp}\exp j[\phi_{j(n+1)}-\phi_{mn}].$$

Equation 15 can be given an interesting physical interpretation. As we shall prove in a moment, the first term gives the contribution to the first-order volume flow from an equivalent simple source of the kind envisioned by Helmholtz and later quantified by Cremer and Bechert. The second term gives the contribution from a fluctuating pressure gradient, due to shear, of the kind suggested vaguely by Rayleigh and used as a model by Coltman. The third term involves contributions to the first order flow due to spectral intermodulation. Thus it appears that the explanations proposed by Cremer and associates and by Coltman are complementary rather than contradictory. Each mechanism contributes to the acoustic drive process in a different way. The third term, representing a purely nonlinear drive mechanism, does not seem to have been previously anticipated.

To establish the above identifications, consider first the partial sound pressure at M due to the first term in Eq. 15:

$$(\mathbf{p}_{m1})_I = -\mathbf{Z}_{m1}(\mathbf{Q}_{m1})_I = [\mathbf{Z}_{m1}\mathbf{Z}_{p1}/(\mathbf{Z}_{s1}+\mathbf{R}_j)]\mathbf{Q}_{j1}. \quad (18)$$

Except for the small term $\mathbf{R}_j$, which seems to represent a sort of intrinsic jet resistance, Eq. 18 is identical with the parallel-resonant circuit representation of Cremer and Ising [See Eq. 1 and Fig. 1(a)]. It may be shown (See Appendix A) that the series impedance, $\mathbf{Z}_{s1}$, is usually quite large in comparison with $\mathbf{R}_j$, even at resonance, so the neglect of $\mathbf{R}_j$ is justified as a first approximation. This term accounts, then, for the partial sound pressure at M due to acoustic condensation, it being proportional to the fluctuating jet volume flow $\mathbf{Q}_{j1}$. The condition $p_m = p_p$ implied by the circuit of Fig. 1(a) can never be completely realized. Even for the Helmholtz mechanism there is a small pressure gradient due to internal inertia of the control volume.

To interpret the second term in Eq. 15 we write the expression for partial pressure at M in terms of a Coltman-type differential pressure:

$$(\mathbf{p}_{m1})_{II} = -\mathbf{Z}_{m1}(\mathbf{Q}_{m1})_{II} \doteq \Delta\mathbf{p}_1(-\mathbf{Z}_{m1}/\mathbf{Z}_{s1}). \quad (19)$$

By comparison with Eq. 4 we may identify the differential pressure across MP as

$$\Delta\mathbf{p}_1 = \frac{2\rho}{S_p}\{(1-A_{jp})[U_{j0}+\tfrac{1}{2}U_{j2}e^{j(\phi_{j2}-\phi_{j1})}] - A_{mp}[U_{m0}+\tfrac{1}{2}U_{m2}e^{j(\phi_{m2}-\phi_{j1})}]\}\mathbf{Q}_{j1}. \quad (20)$$

For a typical pipe, $A_{jp} \ll 1$ and $A_{mp}U_{m2} < A_{mp}U_{m0} \ll U_{j0}$. Therefore, if we restrict attention to the case for which the jet modulation is nearly sinusoidal (making $U_{j2} \ll U_{j0}$), the expression for $\Delta\mathbf{p}_1$ is greatly simplified:

$$\Delta\mathbf{p}_1 = 2\rho(S_j/S_p)U_{j1}U_{j0}\exp^{[j(\omega t+\phi_{j1})]} \quad (21)$$

For some particular steady-state flow modulation mode of the jet we can write

$$U_{j1} = g_1 U_{j0} \quad (22)$$

(where, in general, the constant g_1 is to be determined by Fourier analysis of U_j), giving finally

$$|\Delta\mathbf{p}_1| = (2g_1)(S_j/S_p)\rho U^2_{j0}, \quad (23)$$

which, to within a constant, is identical with Coltman's formula. This interpretation of the second term in Eq. 15, therefore, sees a circuit which is series-resonant, with the drive being supplied by purely hydrodynamic reaction to a fluctuating shear caused by the jet, as represented in Fig. 1(b). Notice, however, that $\Delta\mathbf{p}_1$ actually goes to zero as $\mathbf{Q}_{j1}$ goes to zero (Eq. 20), so that the zero driving flow condition associated with this circuit cannot be realized under the circumstances considered.

The third term in Eq. 15 is the sum of all higher spectral interaction contributions to the first-order pipe flow that do not contain Q_{j1} explicitly. Although this term can be thought of as simply furnishing a higher-order addition to the series resonant drive term, the analogy holds only for some particular mode of jet modulation. It is perhaps better to think of the third term as a separate type of acoustic source that stems from pure nonlinear interaction.

For the special case of sinusoidal jet modulation, the third term vanishes identically, allowing us to write an unambiguous expression for the complex "acoustic gain" of the system:

$$\mathbf{Q}_{m1}/\mathbf{Q}_{j1} = -\mathbf{Z}_{p1}/(\mathbf{Z}_{s1}+\mathbf{R}_j) + [(2\rho/S_p)/(\mathbf{Z}_{s1}+\mathbf{R}_j)]\{(1-A_{jp})U_{j0} - A_{mp} \times [U_{m0}+\tfrac{1}{2}U_{m2}\exp j(\phi_{m2}-2\phi_{j1})]\}, \quad (24)$$

where, also, we have ignored terms in $\mathbf{U}_m$ higher than $\mathbf{U}_{m2}$. In terms of the total pipe flow, this becomes

$$\mathbf{Q}_{p1}/\mathbf{Q}_{j1} = \mathbf{Q}_{m1}/\mathbf{Q}_{j1}+1 = (\mathbf{Z}_{m1}+\mathbf{R}_j)/(\mathbf{Z}_{s1}+\mathbf{R}_j) + [(2\rho/S_p)/(Z_{s1}+R_j)]\{(1-A_{jp})U_{j0} - A_{mp}[U_{m0}+\tfrac{1}{2}U_{m2}\exp j(\phi_{m2}-2\phi_{j1})]\}. \quad (25)$$

It is worth pointing out that neither of the first two circuit diagrams [Figs. 1(a) and 1(b)] describe the way in which the system actually behaves. In general, the organ pipe oscillator does not lend itself to electrical analogy. However, for the case of sinusoidal jet drive, it is possible to synthesize system performance by adding to the parallel resonant network an effective series impedance $\mathbf{Z}_e$ [as in Fig. 1(c)] given approximately by the expression:

$$\mathbf{Z}_e = (2\rho c/S_p)[-\mathbf{Z}_m/(\mathbf{Z}_{s1}+\mathbf{R}_j)](U_{j0}/c), \quad (26)$$

where c is the speed of sound in air.

The importance of the added element in Fig. 1(c) can be estimated by calculating the ratio of $\mathbf{Z}_e$ to the impedance $\mathbf{Z}_{d1}$ of the parallel network:

$$\mathbf{Z}_e/\mathbf{Z}_{d1} = (2\rho c/S_p)(U_{j0}/c)/\mathbf{Z}_{p1}. \quad (27)$$

Near resonance,

$$|\mathbf{Z}_{p1}| \approx |\mathbf{Z}_m| \approx (\rho c/S_p)(k\Delta L), \quad (28)$$

where ΔL is the mouth end correction of the pipe and k is the acoustic propagation constant. Therefore, the magnitude of the ratio near resonance is given by

$$|\mathbf{Z}_e/\mathbf{Z}_{d1}| = (2U_{j0}/c)(k\Delta L)^{-1}. \quad (29)$$

Typical open pipes have mouth end corrections in the neighborhood of one-tenth of the pipe length, while the wavelength at the fundamental is roughly twice the length. Thus the term $k\Delta L$ is of the order $\pi/10$ or about 0.3. Although jet Mach number tends to increase with the design frequency of the pipe, the value of $2U_{j0}/c = 0.053$ for the experimental pipe used by Ising[18] may be considered typical. This leads to an estimated value for the ratio of 0.20 or less. The parallel network is probably dominant near resonance, therefore. However, Ising's investigation showed that the optimum driving point for a pipe is *above* resonance. For this case the larger jet Mach number and smaller pipe impedance both conspire to increase the effectiveness of $\mathbf{Z}_e$. It thus appears that both types of source terms must be taken into account in general.

The relative importance of the third term in Eq. 15 for nonsinusoidal excitation depends on the way in which the jet is modulated. There are two special cases of interest. If the jet edge is symmetrically placed, then we may assume "square wave" modulation.[16] For a very asymmetrically placed jet edge, as in diapason and string-toned pipes, "delta function" drive will result.[25] A rough estimate of the ratio of the third to the second component of $\mathbf{Q}_{m1}$ is possible for these cases. Keeping only the leading term in the summation,

$$|(\mathbf{Q}_{m1})_{III}/(\mathbf{Q}_{m1})_{II}| = \tfrac{1}{2}(U_{j2}/U_{j0})(U_{j3}/U_{j1}). \quad (30)$$

For square-wave modulation, U_{j2} vanishes and the remaining effect would have to come from higher terms in the series. For a diapason pipe, on the other hand, $U_{j3} = U_{j2} = U_{j1} = 2U_{j0}$, so that $(Q_{m1})_{III} \doteq (Q_{m1})_{II}$. Therefore, we conclude that for pipes voiced for symmetrical jet edge, the interaction term will be small, whereas for strongly asymmetrical voicing, the interaction term will have to be considered whenever the type-II source is important.

An additional significance of the type-III source term is that it provides a means for explaining the coupling effects between modes. Using a pipe with modal structure perturbed from its natural state, Benade[26] has shown that the spectrum of a blown pipe cannot be explained via simple passive superposition theory. Although the effects are not so bizarre as those noted by him[23] for strongly nonlinear reed instruments (where the ability to excite a mode at all may depend on the impedance at an upper mode frequency), yet there is clearly a spectral coupling effect indicated for flue pipes. The type-III source term provides one of the nonlinear mechanisms responsible for such phenomena.

D. Second- and Higher-Order Terms

Extracting terms of second order from Eq. 13 and solving for the complex volume flow rate at M yields

$$\mathbf{Q}_{m2} = (\mathbf{Q}_{m2})_I + (\mathbf{Q}_{m2})_{II} + (\mathbf{Q}_{m2})_{III}, \quad (31)$$

where

$$(\mathbf{Q}_{m2})_I = [-\mathbf{Z}_{p2}/(\mathbf{Z}_{s2}+\mathbf{R}_j')]\mathbf{Q}_{j2}, \quad (32a)$$

$$(\mathbf{Q}_{m2})_{II} = [(2\rho/S_p)/(\mathbf{Z}_{s2}+\mathbf{R}_j')]\{(1-A_{jp})[U_{j0}+\tfrac{1}{2}U_{j4}\exp j(\phi_{j4}-2\phi_{j2})] - A_{mp}[U_{m0}+\tfrac{1}{2}U_{m4}\exp j(\phi_{m4}-2\phi_{j2})]\}\mathbf{Q}_{j2}, \quad (32b)$$

$$(\mathbf{Q}_{m2})_{III} = [(\rho A_{jp}e^{j2\omega t})/(\mathbf{Z}_{s2}+\mathbf{R}_j')]\{(1-A_{jp})[\tfrac{1}{2}U_{j1}{}^2e^{j2\phi_{j1}}+U_{j1}U_{j3}\exp j(\phi_{j3}-\phi_{j1})] + A_{mj}(1-A_{mp})[\tfrac{1}{2}U_{m1}{}^2e^{j2\phi_{m1}}+U_{m1}U_{m3}\exp j(\phi_{m3}-\phi_{m1})] - A_{mp}[U_{j1}U_{m1}\exp j(\phi_{j1}+\phi_{m1}) + U_{j1}U_{m3}\exp j(\phi_{m3}-\phi_{j1}) + U_{j3}U_{m1}\exp j(\phi_{j3}-\phi_{m1})]\} + \sum_{n=3} [\rho A_{jp}e^{j2\omega t}/(\mathbf{Z}_{s2}+\mathbf{R}_j')] \times \{\mathbf{a}'U_{jn}U_{j(n+2)} + \mathbf{b}'U_{mn}U_{m(n+2)} + \mathbf{c}'U_{jn}U_{m(n+2)} + \mathbf{d}'U_{j(n+2)}U_{mn}\}, \quad (32c)$$

and where

$$\mathbf{Z}_{s2} = \mathbf{Z}_{m2} + \mathbf{Z}_{p2},$$

$$\mathbf{Z}_{p2} = \mathbf{Z}_{p2}' + j2\rho ck\Delta x/S_p,$$

$$\mathbf{R}_j' = (2\rho A_{jp}/S_p)\{U_{j0}+\tfrac{1}{2}U_{j4} \times \exp j(\phi_{j4}-2\phi_{m2}) - [(1-A_{mp})/A_{jp}] \times [U_{m0}+\tfrac{1}{2}U_{m4}\exp j(\phi_{m4}-2\phi_{m2})]\}, \quad (33)$$

$$\mathbf{a}' = (1-A_{jp})\exp j[\phi_{j(n+2)}-\phi_{jn}],$$

$$\mathbf{b}' = A_{mj}(1-A_{mp})\exp j[\phi_{m(n+2)}-\phi_{mn}],$$

$$\mathbf{c}' = -A_{mp}\exp j[\phi_{m(n+2)}-\phi_{jn}],$$

$$\mathbf{d}' = -A_{mp}\exp j[\phi_{j(n+2)}-\phi_{mn}].$$

It may be verified by inspection that the same three types of sources occur for the second order as for the fundamental field (compare Eqs. 15, 16, and 17). As a matter of notational convenience, the type-III interaction terms involving components of order lower than two were separated from the general summation in Eq. 32c. The presence of the series impedance in the denominator of each term guarantees that $\mathbf{Q}_{m2}$ will be favored for a driver frequency in the vicinity of the second longitudinal pipe mode. Since pipe mode frequencies are normally "stretched," however, and the frequency of $\mathbf{Q}_{j2}$ is necessarily an integral multiple of that for $\mathbf{Q}_{j1}$, it is virtually impossible for both modes to be excited to maximum at once. As blowing pressure is increased (causing the jet speed to rise), there is a

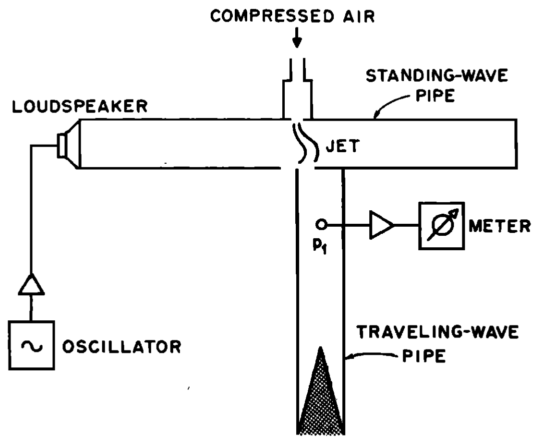

FIG. 3. Schematic of apparatus used by Ising for measuring the amount of fluctuating jet and entrained air flow as a function of jet velocity. [From H. Ising.[18]]

tendency for the amplitude of second harmonic to increase at the expense of fundamental for a pipe blown above its lowest mode. As Elder[25] has shown, some diapasons are voiced to make the second harmonic nearly in tune at the second pipe mode, resulting in a second harmonic almost as strong as the fundamental. Since the higher-order components of $\mathbf{Q}_m$ follow equations similar to Eqs. 31, 32, and 33, we may generalize by saying that, for asymmetrical jet-edge voicing, the character of the spectrum of U_m is primarily determined by the relative nearness of driver harmonics and pipe modes. This is in accordance with the intuitive "recipe" concept of tone spectra suggested by Benade[27] some years ago. Allowance must be made, however, for a certain amount of nonlinear coupling between modes as pointed out by Benade in a later paper.[26]

For the special case of sinusoidal excitation, $\mathbf{Q}_{m2}$ does not vanish, thanks to the type-III source term. This depends not only on U_{j1} but on components of $\mathbf{U}_m$ of order both lower and higher than two (See Eq. 32c), and these in turn depend on still higher-order components. It is clear, therefore, that even with ideal sinusoidal excitation, the spectrum of pipe sound must become increasingly rich with amplitude. For pipes voiced asymmetrically to enhance the nonsinusoidal character of the jet drive, the radiated spectrum is very broad indeed. Boner[28] found significant energy components out to the 30th harmonic for string-toned pipes, and the 20th for diapasons. There is, of course a practical limit to the jet-drive spectrum due to the finite rate at which the jet is capable of crossing into the pipe. The maximum switching rate for the jet is doubtless dependent, in turn, on the scale of the pipe, via feedback in the jet modulation. From the character of the diapason spectrum we may infer that the time spent by the jet in crossing the edge is on the order of a 20th of a period.

II. EFFECT OF ENTRAINED AIR

Returning to the equation for the fundamental, an alternative interpretation of the second term in Eq. 15 is that it represents an additional volume flow, entrained by the jet in the mixing process, and therefore an enhancement of the simple source drive. This is analogous to the point of view adopted by Cremer and Ising in their 1968 paper.[17] From experiment they became convinced that, although the jet starts out with a volume flow proportional to the initial jet speed V_j, the effective final volume flow is proportional to V_j^2, owing to entrainment effects. They determined empirically the following formula for the relation between $(Q_{j1})_{\text{final}}$ and V_j for the conditions of their particular organ pipe:

$$(Q_{j1})_{\text{final}} = 0.032 S_j V_j^2 \; m^3/s. \tag{34}$$

Since their work is not available in English,[29] we will describe briefly their method. The apparatus, which is illustrated schematically in Fig. 3, employed a jet driven across a symmetrical closed pipe at the center. Opposite to the jet orifice was a flow-dividing edge leading on one side to another section of pipe damped so as to produce traveling waves. The cross pipe was supposed to provide resonant standing-wave conditions similar to those found in the mouth of the experimental pipe, while the traveling-wave pipe provided a means of collecting and measuring the total jet-and-entrained air flow. The fluctuating component of the total collected flow was detected by measuring the sound pressure, p_1, of the traveling wave, and taking $Q_{j1} = (S_p/\rho c)p_1$. The constant, 0.032 s/m, was obtained from the slope of the straight line formed by plotting Q_{j1} vs V_j^2, the value of V_j being estimated from upstream pressure.

Referring now to Eq. 16b, the entrained air flow in our representation appears as an addition, not to the jet stream but directly to the pipe flow. A rough comparison with the empirical formula, Eq. 34, is possible if we neglect all but the first term in the bracket (dropping the area ratio factor) and assume the pipe to be near resonance. The magnitude of the flow enhancement is then approximately

$$(Q_{m1})_{II} = 2g_1 \rho U_{j0}^2 (S_j/S_p)/(R_{s1} + R_{j1}), \tag{35}$$

where R_{s1} is the real part of $\mathbf{Z}_{s1}$, and R_{j1} is the leading term in $\mathbf{R}_j$.

Following Ising,[18] we may relate R_{s1} to the more easily measurable damping constant of the system, which is defined in terms of the "3-dB" points on the power level curve:

$$\delta = \Delta f_{3\text{dB}}/f_{\text{res}}. \tag{36}$$

Since power is proportional to $|Q_{p1}|^2$ for constant $|Q_{j1}|^2$, we may determine the 3-dB points from the equation for acoustic gain. Referring to Eq. 25 (although this equation applies strictly for sinusoidal excitation only) we see that the condition for resonance